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ELECTROMAGNETIC COMMUNICATION LABORATORY
TECHNICAL REPORT NO. 86-7

October 1986

AN ANALYSIS OF THE SPECTRAL ITERATIVE TECHNIQUE FOR ELECTROMAGNETIC SCATTERING FROM INDIVIDUAL AND PERIODIC STRUCTURES

A. F. Peterson

Supported by

Department of the Navy
Office of Naval Research
Contract No. N00014-85-K0619

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UILU-ENG-86-2557

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by

A. F. Peterson

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ANALYSIS OF THE SPECTRAL ITERATIVE TECHNIQUE FOR ELECTROMAGNETIC SCATTERING FROM INDIVIDUAL AND PERIODIC STRUCTURES		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER EMC 86-7; UILU-ENG-86-2557
7. AUTHOR(s) A. F. Peterson		8. CONTRACT OR GRANT NUMBER(s) N00014-85-K0619
9. PERFORMING ORGANIZATION NAME AND ADDRESS Electromagnetic Communication Laboratory Department of Elec. and Computer Engineering University of Illinois, Urbana, Illinois 61801		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Project No. 410
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy, Office of Naval Research 800 N. Quincy St. Arlington, VA 22217		12. REPORT DATE October 1986
		13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Electromagnetic Scattering; Spectral Iterative Technique; Method of Moments; Frequency Selective Surface,

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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TABLE OF CONTENTS

LIST OF TABLES

1. INTRODUCTION

The Spectral Iterative Technique (SIT), as developed by Ko, Mittra, Tsao and Kastner [1], [2], [3], has been applied to a wide class of electromagnetic scattering problems, including periodic structures such as frequency selective surfaces [2], [4]. The method has been extended from its original form to incorporate the convergent conjugate gradient method [4], [5]. However, there appears to be some confusion concerning the method and its relationship to other approaches. For instance, Bojarski has claimed that the procedure is identical to a previous method, called the k-space formulation, originally introduced by him [6]. Sarkar and Arvas state that the SIT solves for 2N unknowns in comparison to other iterative methods which solve for N [7]. A common misconception is that the discretization used within the SIT is somehow more general than the method of moments, in that it does not require a choice of basis and testing functions. Recently, Nyo and Harrington [8] concluded correctly that the SIT discretization requires an implicit choice of basis functions, which is often some type of sinc function. However, Bokhari and Balakrishnan [9] have since stated that the SIT uses piecewise constant basis functions for the unknown, and attempted to extend the SIT to incorporate other basis and testing functions. Because it is felt that these and other attributes of the SIT remain largely misunderstood, the following is an attempt to give a unified and consistent interpretation of the SIT type of discretization when applied to both periodic and non-periodic structures. Some of the conclusions that follow have been previously noted by Nyo and Harrington [8].

The term "Spectral Iterative Technique," as it was originally used [2], [3], embodied three specific attributes. First, it contained a specific scheme for discretizing a convolutional integral equation into a discrete system

(matrix equation) containing discrete convolutional symmetries. Second, it embodied a specific iterative algorithm used to solve the discrete system (although this algorithm has been steadily modified in a continuous effort to improve the convergence). Finally, the implementation of the method involved the use of a fast-Fourier transform (FFT) algorithm to perform some, but not necessarily all, of the discrete convolution operations arising throughout the process. This report will concentrate on the first attribute, i.e., the discretization scheme employed within the SIT.

In order to understand the details of the SIT discretization and the different interpretations that can be applied to it, two fundamental relationships must be established. The first is the equivalence between a Toeplitz matrix operator and a discrete convolution, and the implementation of a discrete convolution with an FFT algorithm. The second is the equivalence of the FFT (and the inverse FFT) of a finite length sequence with the analytical Fourier transform (and inverse Fourier transform) of a discrete, periodic function. These concepts are familiar in the areas of signal and system analyses, and are explained in detail in many texts [10], [11], [12].

2. THE FIRST FUNDAMENTAL RELATIONSHIP

A general discrete convolution is an operation of the form

$$e_m = \sum_{n=0}^{N-1} j_n g_{m-n} \qquad m = 0, 1, \ldots, N-1 \tag{2.1}$$

where e, j and g are sequences of numbers (lower-case letters will be used to denote sequences, functions will be assigned upper-case letters). It is easily verified that Equation (2.1) is equivalent to the matrix equation

$$\begin{bmatrix} g_0 & g_{-1} & g_{-2} & & g_{1-N} \\ g_1 & g_0 & g_{-1} & \cdots & \\ g_2 & g_1 & g_0 & & \\ \vdots & & & & \\ g_{N-1} & & & & g_0 \end{bmatrix} \begin{bmatrix} j_0 \\ j_1 \\ \vdots \\ j_{N-1} \end{bmatrix} = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{bmatrix} \tag{2.2}$$

The NxN matrix depicted in Equation (2.2) is a general Toeplitz matrix. Note that all the elements are described by the (2N-1) elements of the first row and column. Now, there are two types of discrete convolutions of interest, those of the circular and linear varieties. Equation (2.1) represents a circular discrete convolution if the elements of "g" repeat with period of length N so that

$$g_{n-N} = g_n \qquad n = 1, 2, \ldots, N-1 \tag{2.3}$$

If the elements of "g" do not satisfy Equation (2.3), Equation (2.1) is a linear discrete convolution. Any linear discrete convolution of length N can be represented by a circular discrete convolution of length 2N-1 by zero-padding the sequence "j" to a length of 2N-1, extending the summation of Equation (2.1) to length 2N-1, and letting the sequence "g" repeat according to Equation (2.3) in order to fill in the required values. (In this context, "g" represents an infinite periodic sequence with period of duration 2N-1.) It is necessary to convert Equation (2.1) to a circular convolution in order to make use of the FFT, as explained below.

The fast-Fourier transform is an efficient way of implementing the discrete-Fourier transform

$$\tilde{g}_n = \sum_{k=0}^{N-1} g_k e^{-j\frac{2\pi nk}{N}} \qquad n = 0, 1, \ldots, N-1 \tag{2.4}$$

The inverse discrete Fourier transform is defined

$$g_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{g}_n e^{j\frac{2\pi nk}{N}} \qquad k = 0, 1, \ldots, N-1 \tag{2.5}$$

For notational purposes, we use

$$\tilde{g} = FFT_N(g) \tag{2.6}$$

$$g = FFT_N^{-1}(\tilde{g}) \tag{2.7}$$

to denote the discrete Fourier transform pair for an N-length sequence. (Lower-case letters without subscripts represent column vectors.) The discrete convolution theorem [13] states that if Equation (2.1) is a circular discrete convolution of length N, it is equivalent to

$$\tilde{e}_n = \tilde{j}_n \tilde{g}_n \qquad n = 0, 1, \ldots, N-1 \tag{2.8}$$

If Equation (2.1) is a linear discrete convolution, the equivalence holds provided that the sequence "j" is zero-padded to length 2N-1, and the FFT's of Equation (2.8) are of length 2N-1.

Thus, the discrete convolution operation of Equation (2.1) is equivalent to the Toeplitz matrix multiplication of Equation (2.2). Furthermore, either (or both) can be implemented using the FFT and inverse FFT algorithm according to

$$e = FFT_N^{-1} \left\{ FFT_N(j) FFT_N(g) \right\} \tag{2.9}$$

(The product of the two column vectors in Equation (2.9) is an ordinary matrix scalar product, without any complex conjugation introduced.) This establishes the first fundamental equivalence discussed above, namely, that an N-th order Toeplitz matrix operator can be implemented with the FFT and inverse FFT. If the discrete convolution is of the circular type, it can be implemented with FFT's of length N; if it is of the linear type, it requires FFT's of length 2N-1.

The above remarks are easily generalized to two or three dimensions. A two-dimensional discrete convolution is an operation of the form

$$e_{pq} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} j_{nm} g_{p-n,\, q-m} \qquad \left\{ \begin{matrix} p \\ q \end{matrix} \right\} = 0, 1, \ldots, N-1 \tag{2.10}$$

This operation is equivalent to the matrix operation

$$\begin{bmatrix} \underline{\underline{G}}_0 & \underline{\underline{G}}_{-1} & \underline{\underline{G}}_{-2} & \cdots & \underline{\underline{G}}_{1-N} \\ \underline{\underline{G}}_1 & \underline{\underline{G}}_0 & \underline{\underline{G}}_{-1} & & \\ \underline{\underline{G}}_2 & \underline{\underline{G}}_1 & \underline{\underline{G}}_0 & & \\ \vdots & & & & \vdots \\ \underline{\underline{G}}_{N-1} & & & & \underline{\underline{G}}_0 \end{bmatrix} \begin{bmatrix} \underline{J}_0 \\ \underline{J}_1 \\ \vdots \\ \underline{J}_{N-1} \end{bmatrix} = \begin{bmatrix} \underline{E}_0 \\ \underline{E}_1 \\ \vdots \\ \underline{E}_{N-1} \end{bmatrix} \tag{2.11}$$

where each element of the NxN block Toeplitz matrix of Equation (2.11) is itself an MxM Toeplitz matrix of the form depicted in Equation (2.2). The relationship established in Equation (2.9) can be extended to multidimensional problems in an obvious manner.

3. THE SECOND FUNDAMENTAL RELATIONSHIP

The second fundamental relationship mentioned above is discussed in detail by Brigham [10]. Suppose we have a discrete function of the form

$$A(x) = \sum_{n=-\infty}^{\infty} a_n \delta(x - n\Delta x) \tag{3.1}$$

where the coefficients of the Dirac delta functions repeat in a periodic manner, i.e.,

$$a_{n+N} = a_n \tag{3.2}$$

We define the Fourier transform integral as

$$F\{H(x)\} = \tilde{H}(f) = \int_{-\infty}^{\infty} H(x) e^{-j2\pi fx} dx \tag{3.3}$$

and the inverse transform as

$$F^{-1}\{\tilde{H}(f)\} = H(x) = \int_{-\infty}^{\infty} \tilde{H}(f) e^{j2\pi fx} df \tag{3.4}$$

The Fourier transform of the discrete, periodic function of Equation (3.1) will also be a discrete, periodic function

$$\tilde{A}(f) = \Delta f \sum_{m=-\infty}^{\infty} \tilde{a}_m \delta(f - m\Delta f) \tag{3.5}$$

where

$$\Delta f = \frac{1}{N\Delta x} \tag{3.6}$$

$$\tilde{a}_m = \sum_{n=0}^{N-1} a_n e^{\frac{-j2\pi mn}{N}} \qquad m = 0, 1, \ldots, N-1 \tag{3.7}$$

Observe that the coefficients "$\tilde{a}$" are exactly the numbers that would be obtained by applying the FFT algorithm to one period of the sequence "a." In other words, the FFT algorithm is equivalent to the analytical Fourier transform of discrete, periodic functions. The same is true for the inverse FFT. This is the second fundamental relationship required for an in-depth analysis of the SIT.

The above relationship enables us to model the effects of the FFT algorithm with the analytical Fourier transform, provided that we first convert the functions under study to discrete periodic functions. This is accomplished as follows. Consider the functions

$$S(x) = \sum_{m=-\infty}^{\infty} \delta(x - m\Delta x) \tag{3.8}$$

$$P(x) = \sum_{q=-\infty}^{\infty} \delta(x - q\Delta X) \tag{3.9}$$

and their Fourier transforms

$$\tilde{S}(f) = \Delta F \sum_{m=-\infty}^{\infty} \delta(f - m\Delta F) \tag{3.10}$$

$$\tilde{P}(f) = \Delta f \sum_{q=-\infty}^{\infty} \delta(f - q\Delta f) \tag{3.11}$$

where

$$\Delta F = \frac{1}{\Delta x} \tag{3.12}$$

$$\Delta f = \frac{1}{\Delta X} \tag{3.13}$$

If an arbitrary function of x is multiplied by S(x), it is converted to a discrete function, i.e., sampled at regular intervals Δx. If an arbitrary function is convolved with P(x), where the continuous convolution is defined

$$A(x) * B(x) = \int_{-\infty}^{\infty} A(x')B(x - x')dx', \tag{3.14}$$

the result is a periodic function of x with period ΔX. In the Fourier transform domain, convolution with $\tilde{S}(f)$ produces a periodic function with period ΔF; multiplication with $\tilde{P}(f)$ produces a discrete function sampled at intervals Δf. In practice, the periods and sampling intervals are related by an integer M, so that

$$\Delta X = M\Delta x \tag{3.15}$$

$$\Delta F = M\Delta f \tag{3.16}$$

Based upon the above relationships, it is clear that simply sampling an arbitrary function at N equally spaced points and applying the FFT to the resulting numbers produces the same result as applying the analytical Fourier transform to the discrete, periodic function created from the same N coefficients. The following sections use the preceding equations to convert continuous functions to discrete, periodic functions in order to model the effects of the FFT and inverse FFT when used in this manner.

4. THE IDEA BEHIND THE ORIGINAL SPECTRAL ITERATIVE TECHNIQUE

The basic idea behind the original Spectral Iterative Technique (SIT) involved the approximation of a continuous Fourier transform using the FFT algorithm, within an iterative numerical solution of a convolutional integral equation. Expressed in one dimension, the convolutional integral equation takes the form

$$E(x) = \int_a^b J(x')K(x - x')dx' \qquad a < x < b \tag{4.1}$$

where $J(x)$ represents the unknown and $E(x)$ and $K(x)$ are given. Note that Equation (4.1) is only valid over the range (a,b). If the equation were valid over $(-\infty,\infty)$, it could be written

$$E(x) = J(x) * K(x) \qquad -\infty < x < \infty \tag{4.2}$$

Using the Fourier transform defined in Equations (3.3) and (3.4) and the convolution theorem [12], Equation (4.2) can be expressed

$$\tilde{E}(f) = \tilde{J}(f)\tilde{K}(f) \tag{4.3}$$

Under suitable restrictions on the analyticity of $\tilde{E}$ and $\tilde{K}$, the solution is given by [14]

$$J(x) = F^{-1}\left\{\frac{\tilde{E}(f)}{\tilde{K}(f)}\right\} \tag{4.4}$$

Unfortunately, the equality in Equation (4.2) does not hold outside the range (a,b). Thus, the above solution process is not applicable in this case.

Although the above procedure is not valid for this example, it illustrates the advantage of working with the Fourier transforms of J(x) and K(x), in order to avoid the convolution of Equation (4.2) in favor of the multiplication of Equation (4.3). This concept can not be implemented directly, as stated above, but it can form the basis of an iterative solution process. In other words, the solution of Equation (4.1) could be obtained in an iterative manner, beginning with an initial estimate of the unknown J(x) and using Equation (4.3) whenever necessary to perform the convolution operation. Although this can be posed as an analytical technique involving the continuous Fourier transform, in practice this would be a numerical method involving the FFT to approximate the Fourier transform. Thus, the process now requires the discretization of the range and domain spaces of the integral operator. The idea is to express the operator in the form

$$\int_a^b J(x')K(x - x')dx' \cong FFT_N^{-1} \{FFT_N(j)\tilde{g}\} \tag{4.5}$$

where the sequence "j" contains coefficients representing the unknown function J(x) and the sequence "$\tilde{g}$" is a discrete representation of the transform $\tilde{K}(f)$. We will return to the construction of "$\tilde{g}$" shortly; in fact, most of the remainder of this report will include an evaluation of different approaches for constructing the sequence. But first, consider Equation (4.5) in the context of some of the preceding discussion. Because we have discretized the continuous operator so that it now maps a sequence of length N to another sequence of length N, it can be represented by an NxN matrix. In addition, it is clear from the first fundamental equivalence relationship established in Section 2 that the operation described by Equation (4.5) is identical in form to that of Equation (2.9), and thus describes a circular discrete convolution of the sequences "j" and "g," where "g" is the inverse FFT of the sequence "$\tilde{g}$." We can conclude from

this that the discretization used within the SIT converted the original convolutional integral equation to a matrix equation having the Toeplitz structure illustrated in Equation (2.2). Note that the primary objective of the SIT is not to produce an explicit NxN matrix to be solved directly but rather to solve the discrete system iteratively, exploiting the FFT algorithm by using the form of the discrete operator given in Equation (4.5).

The basic idea that a discrete system with the Toeplitz type of structure can be solved iteratively using the FFT algorithm to implement the matrix operator is also the basis of the "k-space" method of Bojarski [15], which was developed prior to the SIT. However, the discretization used within the SIT is different from that employed by Bojarski, because of the manner in which the sequence "$\tilde{g}$" is constructed. Bojarski constructed "$\tilde{g}$" by applying the FFT to a sequence "g" obtained directly from the kernel K(x). The discretization used with the SIT required the sequence "$\tilde{g}$" to be constructed by sampling the transform $\tilde{K}(f)$. As stated above, these two methods are not equivalent, because the FFT and the continuous Fourier transform are equivalent only if applied to functions that are discrete and periodic. These functions do not satisfy these criteria. It is worth noting that Bojarski specifically recommended against the approach later used within the context of SIT to construct the sequence "$\tilde{g}$" [16]. In order to interpret the difference between these two approaches, the following section considers the conventional discretization procedure (a generalization of the approach originally used by Bojarski, now known as the discrete-convolutional method of moments [17], [18]) and compares this with the approach used within the SIT.

The above overview attempts to motivate the SIT type of approach. It should be clear that any matrix equation with this type of structure can be solved iteratively in an identical manner, using the FFT to perform the

discrete-convolutional operator. The primary distinction between the SIT approach and others that work with matrix equations having similar symmetries is the manner in which the sequence "$\tilde{g}$" is constructed. In other words, the unique feature of the SIT is the sampling process that produces the sequence "$\tilde{g}$" directly from the analytical Fourier transform $\tilde{K}(f)$. In some cases, specifically those involving periodic geometries, this approach can simplify the problem formulation. In situations where the Fourier transform of the kernel $K(x)$ is much more convenient for numerical calculation than $K(x)$ itself, the SIT approach can be computationally advantageous. In other cases (i.e., non-periodic geometries in free space) the SIT approach can be more difficult to implement in a "correct" fashion. These issues will be investigated in the context of specific examples in the remainder of this report.

5. A COMPARISON OF THE MOMENT-METHOD AND SIT EQUATIONS

Consider the convolutional integral equation of Equation (4.1). E and K are known over the interval of interest and J is an unknown function to be determined. Equation (4.1) can be used to describe scattering from a strip or wire of constant curvature, and is representative of a variety of other electromagnetic scattering problems. A discretization of Equation (4.1) according to the moment-method procedure requires that J be replaced by a finite expansion of the form

$$J(x) \approx \sum_{n=1}^{N} j_n B_n(x) \tag{5.1}$$

where the $\{B_n(x)\}$ are known basis functions and the j_n unknown coefficients. If the expansion is substituted into Equation (4.1) and the resulting equation is made orthogonal to N independent testing functions $\{T_m(x)\}$, the result is a matrix equation of the form

$$e_m = \sum_{n=1}^{N} j_n g_{m,n} \tag{5.2}$$

where

$$e_m = \int_a^b T_m(x) E(x) dx \tag{5.3}$$

and

$$g_{m,n} = \int_a^b T_m(x) \int_a^b B_n(x') K(x,x') dx' dx \tag{5.4}$$

In the general case, g_{mn} represents a fully populated matrix whose NxN entries satisfy no symmetry or redundancy condition.

If the choice of basis and testing functions is restricted to the form

$$B_n(x) = B(x - x_n) \tag{5.5}$$

$$T_m(x) = T(x - x_m) \tag{5.6}$$

where

$$x_n = x_0 + n\Delta x \tag{5.7}$$

and if the basis and testing functions do not overlap the endpoints of the interval (a,b), the discrete system described in Equation (5.2) can be written as

$$e_m = \sum_{n=1}^{N} j_n g_{m-n} \tag{5.8}$$

which is exactly the discrete convolution form discussed in Section 2. Because of this result, the moment method application embodied in Equations (5.5) - (5.8) is denoted the discrete-convolutional method of moments, after Nyo and Harrington [17].

The discretization of Equation (4.1) according to the SIT procedure requires the direct sampling of the Fourier transform $\tilde{K}(f)$. In order for the FFT to be used within the approach, the problem must be expressed as a circular discrete convolution. Equation (5.8) represents a circular discrete convolution only if Equation (4.1) represents a periodic problem, such as a frequency selective surface [2], [4], and then only if the summation is extended over the entire period. Thus, zero padding must be incorporated into the process.

For the moment, consider the nonperiodic case. The simplest approach used within the context of SIT is to directly sample the function $\tilde{K}(f)$ at the required discrete values of f, and use these numbers for the sequence "$\tilde{g}$." For purpose of analysis, we can express this process in general fashion in terms of a discrete, periodic "spectral Green's function" of the form

$$\tilde{G}_{SIT}(f) = \tilde{S}(f) * [\tilde{P}(f)\tilde{W}(f)\tilde{K}(f)] \tag{5.9}$$

where $\tilde{K}(f)$ represents the analytical Fourier transform of K(x), $\tilde{S}(f)$ and $\tilde{P}(f)$ are defined in Equations (3.10) and (3.11) respectively, and $\tilde{W}(f)$ is a function used to window in the transform domain. In the simplest case, the windowing function does nothing more than truncate the function $\tilde{K}(f)$ to one period of the periodic comb function $\tilde{S}(f)$, so that no overlap is introduced by the convolution with $\tilde{S}(f)$. As we will see below, however, the windowing function is an important variable in the process, and other choices for $\tilde{W}(f)$ might be desirable.

During each iteration step, the normal implementation of the SIT requires the multiplication of the sequence "$\tilde{j}$" (constructed from the FFT of the sequence "j" representing the unknown function J(x) of Equation (4.1)) with the sampled values of $\tilde{G}_{SIT}(f)$. The inverse FFT is then applied to transform the sequence back to the spatial domain. Because of the fundamental relationships of Sections 2 and 3, we know that this is equivalent to the Toeplitz matrix multiplication presented in Equation (2.2). Furthermore, we can explicitly construct the matrix elements by modeling the inverse FFT of the sequence "$\tilde{g}$" with the formulas presented in Section 3. Since $\tilde{S}(f)$ and $\tilde{P}(f)$ are comb functions, this can be expressed in terms of the discrete function

$$G_{SIT}(x) = P(x) * [S(x)\{W(x) * K(x)\}] \tag{5.10}$$

The matrix elements from Equation (5.10) are given by

$$g_{m-n}^{SIT} = \sum_{q=-\infty}^{\infty} \left\{ W(x) * K(x) \Big|_{x = (m-n)\Delta x - q\Delta X} \right\} \tag{5.11}$$

If the moment-method process described in Equations (5.1) to (5.8) is generalized to produce an infinite-periodic sequence for comparison to Equation (5.10), the result is

$$G_{MM}(x) = P(x) * [S(x)U(x)\{T(-x) * B(x) * K(x)\}] \tag{5.12}$$

or, equivalently,

$$g_{m-n}^{MM} = T(-x) * B(x) * K(x) \Big|_{x = (m-n)\Delta x} \tag{5.13}$$

$B(x)$ is the basis function introduced in Equation (5.5), and $T(-x)$ is a space-reversal of the testing function $T(x)$ appearing in Equation (5.6). The notable difference between the form of Equations (5.10) and (5.12) is the appearance of $U(x)$ in the moment-method function. $U(x)$ is necessary to truncate the spatial kernel $K(x)$ to the period in order to avoid aliasing errors when the fictitious periodicity is introduced through the convolution with $P(x)$. For instance, $U(x)$ may be of the form

$$U(x) = \begin{cases} 1 & x \varepsilon (a,b) \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

Of course, the period may be chosen to be larger than the interval (a,b), and $U(x)$ may vary accordingly. The aliasing errors due to the absence of $U(x)$ are clearly illustrated by the infinite summation in Equation (5.11).

There are several conclusions that can be drawn from the above comparison. Before expanding on these, it is worth noting that the period size is a parameter to be selected by the user. For nonperiodic problems of the type described above, the SIT approach according to Equation (5.9) actually involves the approximation of a single scatterer by a periodic array of scatterers. Thus, the period size must be initially very large in order to accurately model a single scatterer. (Once the sequence "g" is constructed in the spatial domain via the inverse FFT, it can be truncated to a much smaller size before the computationally intensive iteration process begins.) In the limit as the period size approaches infinity, the discrete systems constructed by the SIT and the moment method are equivalent provided that

$$W(x) = T(-x) * B(x) \tag{5.15}$$

This fact was recently noted by Nyo and Harrington [8].

In view of the above comparison, the SIT discretization could be generalized to incorporate a function corresponding to the U(x) used with the moment method. This could greatly reduce the array sizes and initial computation required to implement the SIT. However, $\tilde{U}(f)$ appears within a convolution in the spectral domain, and the desired $\tilde{U}(f)$ (the transform of Equation (5.14)) is a so-called sinc function

$$\tilde{U}(f) = \frac{\sin(\pi f/\Delta f)}{\pi f} \tag{5.16}$$

with support over the entire x-axis. Because of this, in general it is difficult to include the convolution with $\tilde{U}(f)$ in a numerical implementation.

There appear to be two ways in which the effects of U(x) could be included approximately in the SIT procedure. The first is simply to extend the period to

some large interval, and approximate the transform $\tilde{U}(f)$ by a Dirac delta function (which it approaches as the period becomes sufficiently large). This is the technique used in the literature [1], [3], [5]. (Note that a given function is not altered after convolution with a delta function, and thus Equation (5.9) suffices to describe the process.) An alternate approach is to approximate the rectangular truncation function of Equation (5.14) with a smoother function, in order to obtain a $\tilde{U}_{ap}(f)$ with finite support (at least approximately). The smoother $U_{ap}(x)$ must be sufficiently flat over the spatial interval of interest, in order to avoid distorting the desired spatial Green's function, yet yield a transform which can be conveniently included in the convolution operation of the generalized discrete spectral Green's function

$$\tilde{G}_{SIT}(f) = \tilde{S}(f) * \tilde{U}_{ap}(f) * [\tilde{P}(f)\tilde{W}(f)\tilde{K}(f)] \tag{5.17}$$

An additional advantage of the second approach is that a singularity often present in $\tilde{K}(f)$ is explicitly smoothed by convolution with $\tilde{U}_{ap}(f)$.

Because the windowing function $\tilde{W}(f)$ appears as a multiplication with $\tilde{K}(f)$, the SIT approach can easily incorporate a variety of windowing functions. Based upon the comparison with the moment method, it appears that a primary consideration for the choice of $\tilde{W}(f)$ should be the corresponding spatial domain basis function selected implicitly in the process. For instance, the choice of a rectangular window for $\tilde{W}(f)$ corresponds to the implicit choice of a sinc function for the basis function. Since sinc functions have unbounded support, they do not appear to be appropriate approximations to subsectional basis functions, and will apparently have considerable support outside the original domain of interest (i.e., outside the original scatterer). Typical subsectional basis functions, such as a piecewise constant or a triangle function, thus correspond to windowing functions $\tilde{W}(f)$ with unbounded support, which seems to suggest that the

"proper" windowing function to use with the SIT is one which allows considerable aliasing in the spectral domain. Thus, the incorporation of a windowing function that corresponds to a subsectional basis function may be complicated by the need to deliberately overlap many periods of the function $\tilde{K}(f)$ when constructing $\tilde{G}_{SIT}(f)$.

By analogy with the moment-method procedure, it is obvious that a testing function could be incorporated into the SIT process, as may be necessary if the excitation in a given problem is highly localized. The choice of $\tilde{W}(f)$ can be made to correspond to both a basis and a testing function, as indicated by Equation (5.15). The excitation sequence "e" can be computed according to Equation (5.3).

Up until now, we have considered only nonperiodic problems. If Equation (4.1) represents a periodic problem, it can be discretized with the SIT approach without the detrimental effects introduced by the periodic nature of the FFT algorithm. Since the Fourier transform of a periodic function is discrete, Equation (5.9) simplifies to

$$\tilde{G}_{SIT}(f) = \tilde{S}(f) * [\tilde{W}(f)\tilde{K}_p(f)] \tag{5.18}$$

and Equation (5.11) is given by

$$g_{m-n}^{SIT} = W(x) * K_p(x)\Big|_{x = (m - n)\Delta x} \tag{5.19}$$

Thus, the SIT process and the moment-method process produce equivalent discrete systems for the periodic case as long as the windowing function is chosen to satisfy Equation (5.15). Since the function $K_p(x)$ is normally a slowly convergent infinite summation in the periodic case, the SIT process might be far

more efficient than the conventional moment method from a computational standpoint.

To summarize, two discretization procedures have been outlined. The first, the discrete-convolutional method of moments, involves the explicit introduction of basis and testing functions in the spatial domain. The second, the SIT type of discretization, involves sampling the Fourier transform of the kernel or Green's function in anticipation of the use of the inverse FFT algorithm to return to the spatial domain. (In practice, the sequence "$\tilde{g}$" is used directly in the transform domain; the inverse FFT is present in the analysis because of the first fundamental relationship established in Section 2.) By modeling the FFT algorithm with the functions introduced in Equations (3.8) - (3.13), an equivalence is made between the SIT and moment-method discretizations. It follows that the SIT approach involves implicit basis/testing functions introduced through the windowing function $\tilde{W}(f)$.

The easiest way to implement the SIT is to directly sample $\tilde{K}(f)$ at the desired discrete values of f. This approach is equivalent to the use of a rectangular window $\tilde{W}(f)$, and produces a spatial domain matrix equation that could have been obtained from sinc basis functions (to be more precise, the convolutions of the implicit basis and testing functions are sinc functions). This simple approach appears to be the technique consistently used in the literature [1] - [5]. Here, we have suggested an "extended" form of the SIT, which permits the incorportion of any basis and testing functions into the process. If it is desired to incorporate subsectional basis functions, Equation (5.9) will involve an infinite summation.

With regard to the SIT, the above remarks assume that the spatial period is taken to correspond to the true period (in the case of a periodic structure) or

to be much larger than the scatterer and the wavelength (in the case of an individual scatterer). In addition, an appropriate amount of zero-padding must be incorporated into the process. The SIT discretization introduces a fictitious periodicity into the modeling process for individual scatterers, and this effect is examined in the following section for the example of scattering from an individual strip.

6. TM-WAVE SCATTERING FROM A STRIP

As an example of the implementation of the moment-method and the SIT procedures, and a means to compare the two to develop guidelines for the use of the latter approach, consider the problem of TM-wave scattering by a perfectly conducting flat strip. The integral equation for a one-wavelength strip has the form

$$E(x) = \int_{-0.05}^{0.95} J(x')K(x - x')dx' \qquad -0.05 < x < 0.95 \tag{6.1}$$

The kernel in this case is given by

$$K(x) = \frac{1}{4j} H_0^{(2)}(2\pi|x|) \tag{6.2}$$

and its Fourier transform is

$$\tilde{K}(f) = \begin{cases} \dfrac{1}{j4\pi\sqrt{1 - f^2}} & |f| < 1 \\ \dfrac{1}{4\pi\sqrt{f^2 - 1}} & |f| > 1 \end{cases} \tag{6.3}$$

If ten basis functions are used with the moment-method procedure, specifically piecewise constant or "pulse" functions defined by

$$B(x) = \begin{cases} 1 & x\epsilon\ (-0.05,\ 0.05) \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

and if the testing functions are Dirac delta functions

$$T(x) = \delta(x) \tag{6.5}$$

where

$$x_n = -0.1 + n(0.1) \qquad n = 1, 2, \ldots, 10 \tag{6.6}$$

then the numerical values of the spatial domain sequence representing "g" are given in Table I. This sequence was computed from Equation (5.4), and is equivalent to the values of the discrete spatial Green's function $G_{MM}(x)$ described in Equations (5.12) or (5.13).

Consider an SIT discretization of Equation (6.1), assuming that the function $\tilde{U}_{ap}(f)$ is taken to be a Dirac delta function as discussed in Section 5. A rectangular window is used for $\tilde{W}(f)$, which means that we are specifically choosing alternate basis functions than those employed in the moment-method system above. Thus, the data from Table I are not the numerical values which would be produced by the SIT procedure even if an infinite amount of zero-padding was incorporated into the process, because $B(x)$ and $\tilde{W}(f)$ are not a transform pair. However, this choice for $\tilde{U}_{ap}(f)$ and $\tilde{W}(f)$ is the easiest to incorporate into the SIT approach, and appears to be the approach used in the literature [1] - [5].

Table II shows the values of the sequence "g" produced by the SIT procedure for a period length of 102.3 wavelengths (M=1023). This table also shows the relative difference between the data of Table I and the SIT data. In spite of the fact that we do not expect perfect agreement, the numerical values are similar. From a study using a variety of strip sizes, and incorporating a transform pair for the basis function and windowing function, it appears that the equivalent spatial period must exceed 25 wavelengths in order to obtain agreement within five percent (in the first few values of the sequence "g") between the moment-method and SIT numbers. The period must exceed 100 wavelengths if the

TABLE I

DISCRETE SPATIAL DOMAIN SEQUENCE PRODUCED BY MOMENT METHOD WITH PULSE BASIS FUNCTIONS AND DIRAC DELTA TESTING FUNCTIONS FOR A STRIP OF LENGTH 1 λ WITH 10 CELLS.

n	$\lvert g \rvert$	$\angle g$ (degrees)
0	0.0436	-34.67
1	0.0237	-71.33
2	0.0171	-111.38
3	0.0141	-149.03
4	0.0123	174.06
5	0.0110	137.49
6	0.0101	101.10
7	0.0093	64.81
8	0.0087	28.59
9	0.0082	-7.58

TABLE II

DISCRETE SPATIAL SEQUENCE PRODUCED BY THE SIT USING A RECTANGULAR WINDOW $\tilde{W}(f)$ WITH M=1023.

n	$\lvert g \rvert$	$\angle g$ (degrees)	% diff. as compared to Table I
0	0.0433	-35.27	1.2 %
1	0.0237	-72.46	2.0 %
2	0.0176	-114.36	6.0 %
3	0.0139	-148.46	2.0 %
4	0.0117	173.20	5.2 %
5	0.0104	132.65	10 %
6	0.0101	96.03	8.9 %
7	0.0097	62.57	5.3 %
8	0.0085	29.60	2.7 %
9	0.0075	-8.72	9.2 %

first few values of the sequence are to agree within one percent [18]. In general, it appears that the difference between the moment method and SIT sequence is primarily due to the fictitious periodic nature of the SIT representation, and not the difference in basis functions. In other words, the SIT sequence based upon the implicit basis function

$$W(x) = \frac{\sin(\pi x/\Delta x)}{(\pi x/\Delta x)} \tag{6.7}$$

agrees well with the sequence produced by the moment method with pulse basis functions (assuming the aliasing effects are suppressed), at least for this example. Since it appears that most of the previous results obtained with the SIT used implicit basis functions of the form of Equation (6.7), this may explain the reported success of the procedure [1] - [5], [9].

7. SCATTERING FROM A PERIODIC STRIP GRATING

Consider the problem of scattering of a plane wave from an infinite, periodic strip grating. The integral equation has the form

$$E(x) = \int_{-0.05}^{0.95} J(x')K_p(x - x')dx' \qquad -0.05 < x < 0.95 \tag{7.1}$$

where the strip widths are taken to be one wavelength. The difference between Equation (7.1) and (6.1) is the kernel

$$K_p(x) = \frac{1}{4j} \sum_{q=-\infty}^{\infty} H_0^{(2)}(2\pi|x - q\Delta X|)e^{-j\beta q\Delta X} \tag{7.2}$$

where ΔX represents the spatial period, and the parameter β, a function of the incident field, is given by

$$\beta = 2\pi \cos \theta \tag{7.3}$$

for a plane-wave incident field propagating in the θ direction, where $\theta = 0$ is $+\hat{x}$ and $\theta = \pi/2$ is $+\hat{y}$ (assuming the strip lies in the $y = 0$ plane, and is infinite is extent in the z direction). The Fourier transform of the kernel is

$$\tilde{K}_p(f) = \tilde{P}(f + \frac{\beta}{2\pi})\tilde{K}(f) \tag{7.4}$$

where $\tilde{K}(f)$ is the transform of the nonperiodic kernel, as defined in Equation (6.3), and $\tilde{P}(f)$ is defined in Equation (3.11).

Supposing that the spatial period is 1.5 wavelengths, and the incident field is normally incident upon the strip grating, the numbers computed from the conventional moment-method formulation for pulse basis functions and Dirac delta

testing functions are given in Table III. In order to compute these, it is first necessary to find a way of accelerating the convergence of the series of Equation (7.2), as otherwise the summation converges too slowly to be useful. Poisson summation formulas have been tabulated for this purpose [19]. Note that the values appearing in Table III correspond to the definition of Equation (5.13), with $K(x)$ replaced by $K_p(x)$.

The corresponding sequence from the SIT using a rectangular window $\tilde{W}(f)$ is given in Table IV. In this case, the sequence "$\tilde{g}$" is constructed from the formula given in Equation (5.18). Because of the rectangular window employed for this example, the expression simplifies to the form

$$\tilde{g}_n = \begin{cases} \tilde{K}\,(n\Delta f) & n = 0, 1, \ldots, 7 \\ \tilde{K}([15 - n]\Delta f) & n = 8, 9, \ldots, 14 \end{cases} \tag{7.5}$$

where $\tilde{K}(f)$ is defined in Equation (6.3). After "$\tilde{g}$" is computed, the inverse FFT is used to construct the data shown in Table IV. Note that the sequences of Tables III and IV should differ, because the SIT system actually uses sinc basis functions.

We next consider the "correction" of the SIT in order to incorporate pulse basis functions instead of the sinc function. This requires that we use the window

$$\tilde{W}(f) = \frac{\sin\,(\pi f/\Delta f)}{\pi f} \tag{7.6}$$

when constructing the discrete spectral Green's function

$$\tilde{G}_{SIT}(f) = \tilde{S}(f) * [\tilde{W}(f)\tilde{K}_p(f)] \tag{7.7}$$

TABLE III

DISCRETE SPATIAL DOMAIN SEQUENCE PRODUCED BY MOMENT METHOD WITH PULSE BASIS FUNCTIONS AND DIRAC DELTA TESTING FUNCTIONS FOR A PERIODIC STRIP GRATING WITH PERIOD EQUAL TO 1.5 λ AND STRIP SIZE EQUAL TO 1.0 λ, USING 10 CELLS ON THE STRIP. THE INCIDENT FIELD IS A TM PLANE WAVE AT NORMAL INCIDENCE.

n	$\lvert g \rvert$	$\angle g$ (degrees)
0	.0360	-32.70
1	.0184	-80.86
2	.0170	-119.67
3	.0146	-138.67
4	.0089	-154.62
5	.0028	140.96
6	.0073	57.51
7	.0115	48.00
8	.0115	48.00
9	.0073	57.51

Due to the convolution with $\tilde{S}(f)$, these values can be explicitly written as

$$\tilde{g}_n^{SIT} = \Delta F \sum_{m=-\infty}^{\infty} \left\{ \tilde{W}(f)\tilde{K}(f) \Big|_{f = m\Delta F + n\Delta f - \frac{\beta}{2\pi}} \right\} \tag{7.8}$$

Thus, in order to incorporate pulse basis functions into the SIT procedure, we actually must superimpose the contributions from the overlapping functions according to the summation given in Equation (7.8). For the above example, with period of 1.5 wavelengths and a normally incident plane wave, the transform domain sequence must be computed according to

$$\tilde{g}_n^{SIT} = \frac{\Delta F}{4\pi^2} \sum_{m=-\infty}^{\infty} \left\{ \frac{\sin(\pi f/\Delta F)}{f\sqrt{f^2 - 1}} \Bigg|_{f = m\Delta F + n\Delta f} \right\} \tag{7.9}$$

(The proper branch of the square root must be used, as indicated in Equation 6.3.) Table V shows the values of the spatial sequence "g" obtained by this process. In this case, there is perfect agreement between the moment-method sequence from Table III and the SIT sequence from Table V. In fact, the acceleration procedure used in the construction of Table III requires the Green's function to be Fourier transformed and exchanges the summation in the spatial domain with the explicit inverse Fourier transformation (which is itself a summation in this case, but faster converging). The basic difference in the construction of Tables III and V is that the FFT algorithm is used explicitly to compute the inverse transformation of "$\tilde{g}$" for Table V. Here, the functions are discrete and periodic; thus, the FFT and Fourier transforms are equivalent.

We have investigated the difference between the discrete systems produced by the SIT and moment method for the example of TM-wave scattering by a periodic

TABLE V

DISCRETE SPATIAL DOMAIN SEQUENCE PRODUCED BY THE EXTENDED SIT IN ORDER TO USE IMPLICIT PULSE BASIS FUNCTIONS AND DIRAC DELTA TESTING FUNCTIONS, FOR A PERIODIC STRIP GRATING WITH PERIOD OF 1.5 λ AND A NORMALLY INCIDENT TM PLANE WAVE.

n	$\|g\|$	$\angle g$ (degrees)
0	.0360	-32.70
1	.0184	-80.86
2	.0170	-119.67
3	.0146	-138.67
4	.0089	-154.62
5	.0028	140.96
6	.0073	57.51
7	.0115	48.00
8	.0115	48.00
9	.0073	57.51

strip grating. An "extended" SIT procedure is demonstrated that can be used to produce a sequence "g" that is identical to the sequence produced by the moment method for any choice of basis and testing functions. It turns out that this extension of the SIT is actually equivalent to one of the standard Green's function acceleration techniques. In previous work [1] - [5], a simple rectangular window was often employed within the SIT, because this circumvents the need to sum a series to compute "$\tilde{g}$." The implicit basis functions used in connection with a rectangular window are sinc functions, which suggests that the results for the SIT approach may not be in good agreement with those of the moment method if the latter uses subsectional basis and testing functions. However, for the example of a TM wave incident upon a strip grating, the two approaches produce almost identical numbers. We would not necessarily expect this to be the case for other types of problems, however, especially those involving derivatives on the kernel. For those problems, it may be necessary to use the extended SIT discretization procedure.

8. SUMMARY

An interpretation of the discretization used within the Spectral Iterative Technique (SIT) is presented that attempts to clarify the relative similarities and differences of the SIT and the conventional moment-method approaches. In addition, an extension of the SIT is developed to enable the incorporation of explicit basis and testing functions into the procedure. Both individual scatterers and periodic scatterers are considered, as the method is somewhat different for these two types of problems.

For an integral equation of the form

$$E(x) = \int_a^b J(x')K(x - x')dx' \qquad a < x < b \tag{8.1}$$

the discrete system generated by the SIT is Toeplitz (or for multidimensional problems, block Toeplitz with Toeplitz elements). For integral equations of the form

$$E(x) = R(x)J(x) + \int_a^b J(x')K(x - x')dx' \qquad a < x < b \tag{8.2}$$

the diagonal elements of the system differ from the purely Toeplitz form, but the off-diagonal entries are Toeplitz in structure. Since routines are available for treating Toeplitz systems by direct methods [20] - [22], it may be more efficient to solve the discrete systems directly, regardless of the manner in which the matrix elements are constructed. In other words, it may be more efficient to construct the discrete system using the SIT approach, but then to solve the system using a special Toeplitz routine (instead of an iterative algorithm).

Because of the implicit basis and testing functions introduced by the SIT discretization, there is no additional generality in using the SIT as opposed to the conventional method-of-moments formulation. Numerical results generated for the two examples of Sections 6 and 7 indicate that the SIT systems can be a good approximation to the moment method systems, even if the implicit basis functions appear to be inappropriate for the problem under consideration. Thus, although there is no additional generality to the SIT, the method may be computationally favorable to the conventional approaches. This tradeoff in efficiency can only be evaluated in the context of specific problems. For the two examples considered here, there was no clear advantage to the SIT formulation. In fact, for the individual scatterer examined in Section 6, there is clearly appreciable error introduced by the SIT discretization unless the equivalent spatial periods are on the order of 100 wavelengths (not a practical size for anything except one-dimensional structures). Problems best suited to the SIT formulation are periodic structures and any type of problem where the Fourier transform of the kernel is much easier to compute than its spatial domain counterpart.

The above interpretation of the SIT is an expansion of a brief exposition from reference [18]. A similar but less complete analysis was presented previously in reference [8].

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